A New Approach to Evaluate Second Kind of Volterra Integral Equation with Homotopy Perturbation Method and Variational Iteration Method

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Abstract- An analytic technique, combination of two methods Homotopy Perturbation Method and Variational Iteration Method are applied for solve second kind of linear Volteera equations. homotopy perturbation method and variational iteration method are employed to solve Volteera integral equations. The result reveals that the homotopy perturbation method and variational iteration method are very effective and simple. But at the end it is observed that homotopy perturbation method in some of the issues lead to exact solution and variational iteration method lead to limit solution. To illustrate the ability and reliability of the method, three examples are given, revealing its effectiveness and simplicity.

Keywords- homotopy perturbation method, Variational Iteration Method ,Integral Equations, Numerical Method, Second kind of Volterra integral equation.

1. INTRODUCTION

Integral equations find special applicability within scientific and mathematical disciplines. To check the numerical method, it is applied to solve different test problems with known exact solutions and the numerical solutions obtained confirm the validity of the numerical method and suggest that it is an interesting and viable alternative to existing numerical methods for
solving the problem under consideration.
Variational iteration method [10, 11, 12] is a powerful device for solving various kinds of equations, linear or non-linear. Volterra integral equations have been solved by classical numerical and theoretical methods [13,14]. In this paper, these methods are applied for the Volterra integral equations. The general form of these integral equations is given by the corresponding Volterra equations for φ take the form

$$\varphi(x) = F(x) + \lambda \int_a^x K(x,t)\varphi(t)dt \,,$$

Volterra integral equation of 2nd Kind

2. Basic Idea of Homotopy-Perturbation Method

To clarify the basic ideas of the homotopy perturbation method [13], we consider the following nonlinear equation

$$A_u + N_u = 0, \qquad (1)$$

whereAand N are linear operator and non-linear operator, respectively. In order to use the homotopy perturbation, a suitable construction of a homotopy equation is of vital importance. Generally, a homotopy can be constructed in the form

$$A_u + P\,(N_u - A_u + N_u) = 0, \qquad (2)$$

whereL can be a linear operator or a simple non-linear operator, and the solution of $A_u = 0$ with possible some unknown parameter can best describe the original non-linear system.

The non-linear Volterra integral equations are given by

φ (x)=f(x)
$+\int_0^x [k(x,t)\{R(\varphi(t)) + N(\varphi(t))\}dt]$

φ (x) is a unknown function that will be determined, K(x,t) is the kernel of the integral equation, f(x) is an analytic function, R(φ) and N(φ) are linear and nonlinear functions of φ , respectively . To illustrate the HPM, we consider (3) as

$$A(x) = \varphi(x) - f(x) - \int_0^x K(x,t)\{R(\varphi(t)) + N(\varphi(t))\}dt = 0 \qquad (4)$$

As a possible remedy, we can define H(u,p) by

H(φ ,0) = F(φ) , H(φ ,1) = A(φ), (5)

whereF(φ) is an integral operator with known solution u_0, which can be obtained easily. Typically, we may choose a convex homotopy by

H(φ ,p) = (1- p)F(φ) + pA(φ) (6)

and continuously trace an implicitly defined curve from a starting point H(u_0,0) to a solution function H(φ ,1). The embedding parameter p monotonically increase from zero to unit as the trivial problem A(φ) = 0.The embedding parameter p $\in$ [0, 1] can be considered as an expanding parameter.
The embedding parameter p [15] is introduced much more naturally, unaffected by artificial factors. Furthermore, it can be considered as a small parameter for $0 < p \le 1$.
So it is very natural to assume that the solution of (5, 6) can be expressed as

$$\varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + \cdot\cdot\cdot$$

Equating the items with identical power of p, we get

$P^0 : \varphi_0(x) = f(x)$

$$P^1 : \varphi_1(x) = \int_0^x (k(x,t)\varphi_0(t))\,dt\ ,$$

$$P^2 : \varphi_2(x) = \int_0^x (k(x,t)\varphi_1(t))dt\ ,$$

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$$P^n : \varphi_n(x) = \int_0^x (k(x,t)\varphi_{n-1}(t))dt \qquad (7)$$

Therefore, when p=1, the approximate solution of above equation can be readily obtained as follows:

$$\varphi = \lim_{p\to1} u_0 + u_1 + u_2 + \cdot\cdot\cdot\ .$$

The combination of the perturbation method and the homotopy method is called the HPM, whicheliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The above series is convergent for most cases.

3. Variational Iteration Method

Let us suppose

$$\varphi(x) = f(x) + \int_a^x K(x,t)\varphi(t)dt\ ,$$

For solving above equation by VIM, first we differentiate once from both sides the equation with respect to x:

$$\varphi'(x) = f'(x) + k(x,x) + F(\varphi(x)) \int_0^x \frac{\partial k(x,t)}{\partial x} F(\varphi(t))dt$$

Now we apply Variational iteration method to equation (8). According to this method correction functional can be written in the following form:

$\varphi_{n+1}(x) = \varphi_n(x)$
+
∫↓0↑x≅〖λ 〖 〖[(φ〗 ↓n〗 ↑, (x) – f↑(,) (x) – k(x, x)F(((u↓n (
]

To make the above correction functional stationary with respect to φ_n, we have:

$\delta_{n+1}(x) = \delta_n(x)$
+
δ〖↓0↑x≅〖λ 〖 〖[(φ〗 ↓n〗 ↑, (s) – f↑(,) (x) – k(x, x)F(((u↓n
]

$$= \delta\varphi_n(x) + \int_0^x \lambda(x)\delta[(\varphi]_n{}'(x))dx = \delta\varphi_n(x) + \lambda(x)\delta\varphi n(x) + \int_0^x \lambda'\delta\varphi n(x)dx = 0$$

From the above relation for any $\delta\varphi_n$, we obtain the Euler-Lagrange equation:

$$\lambda'(x) = 0 \qquad (9)$$

With the following natural boundary condition:

$$\lambda(x) + 1 = 0 \qquad (10)$$

Using equations (9) and (10), Lagrange multiplier can be identified optimally as follows:

$$\lambda(x) = 1 \qquad (11)$$

Substituting the identified Lagrange multiplier into equation, we obtain the following iterative relation:

$$\varphi_{n+1}(x) = \varphi_n(x) - \left[\int_0^x \left[\left[\varphi_n\right]'(x) - f'(x) - k(x,x)F(\varphi_n(x)) - \int_0^x \frac{\partial k(x,t)}{\partial x} F(\varphi_n(t))\,dt\right]dx\right] \qquad (12)$$

By starting from $\varphi_0(x)$, we can obtain the exact solution or an approximate solution to the equation. Also in some Volterra integral equations by differentiating from integral equation, for example when the kernel is independent of *x*, we obtain a differential equation and we can solve differential equation by variational iteration method.

4. Numerical Examples

Example: 1 Consider the following non-linear Volterra integral equation

$$y(x) = \sec x + \tan x + x - \int_0^x (1 + u^2(t))\,dt \qquad (13)$$

the exact solution $\varphi(x) = \sec x$

First solve by Homotopy perturbation method:

We define

$F(u) = \varphi(x) - \sec x$

$A(u) = \varphi(x) - \sec x - \tan x - x + \int_0^x (1 + u^2(t))\,dt$

and substituting F(u) and A(u) in (6) and equating the terms with identical powerof p , we obtain:

$P^0 : \varphi_0(x) = \sec x$

$P^1 : \varphi_1(x) = -\tan x - x + \int_0^x \left(1 + \varphi_0^2(t)\right)dt = 0$

$P^2 : \varphi_2(x) = -\tan x - x + \int_0^x \left(1 + \varphi_1^2(t)\right)dt = 0$

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$P^{n+2} : \varphi_{n+2}(x) = -\tan x - x + \int_0^x \left(1 + \varphi_{n+1}^2(t)\right)dt = 0$

$\varphi(x) = u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots\ldots\ldots\ldots\ldots = \sec x$

Now solve by Variational Iteration Method:

By differentiating once from integral equation (12), we obtain the following differential equation

$$\varphi'(x) + \varphi^2(x) - \tan x \sec x - \tan^2 x - 1 = 0 \qquad (14)$$

by applying variational iteration method for equation (14), we derive the following iterative formula:

$$\varphi_{n+1}(x) = \varphi_n(x) - \int_0^x \left(\left[\varphi_n\right]'(s) + \varphi_n^2(s) - \tan s \sec s - \tan^2 s - 1\right) ds$$

Consider initial approximation $u_0(x) = \sec x$ and by the iterative formula (15), we get:

$u_1(x) = \sec x$

$u_2(x) = \sec x$

$u_3(x) = \sec x$

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Therefore, the exact solution can be recognized easily.

Example: 2consider the following non-linear Volterra integral equation

$$u(x) = f(x) - \int_0^x \left(e^{-(t-x)}\right) u(t)\,dt \; .$$

where

$f(x) = \cosh x,$

with the exact solution $y(x) = e^{-x}$

First solve by Homotopy perturbation method:

and substituting F(u) and A(u) in (6) and equating the terms with identical power of p , we obtain:

$P^0 : \varphi_0(x) = \cosh x$

$P^1 : \varphi_1(x) = -\int_0^x (e^{-(t-x)} u_0(t)dt) = -\frac{1}{2}\sinh x - \frac{1}{2} x \sinh x - \frac{1}{2}x \cosh x,$

$P^2 : \varphi_2(x) = -\int_0^x (e^{-(t-x)} u_1(t)dt) =$

$-\frac{1}{4}\sinh x + \frac{1}{4} x \sinh x + \frac{1}{4}x^2 \sinh x + \frac{1}{4}x \cosh x + \frac{1}{4}x^2 \cosh x$

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$P^n : \varphi_n(x) = -\int_0^x (e^{-(t-x)} u_{(n-1)}(t)dt)$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x)$$

$$= \frac{1}{2}\sin 2x + \frac{1}{5}\sin x + \frac{589}{5600}\sin x - \frac{9}{160}\sin 3x + \frac{3}{40}x\cos x - \frac{1}{10}\sin 2x + \frac{9}{1400}\sin 6x + \ldots\ldots\ldots\ldots\ldots$$

Now solve by Variational Iteration Method:

The corresponding iterative relation (14) for this example can be constructed as:

$$\varphi_{n+1}(x) = \varphi_n(x) - \int_0^x (\varphi_n'(x)e^{-(t-x)}\varphi(t)dt)dx$$

by taking $u_0(x) = \cosh x$,

we derive the following results:

$u_1(x) = -\frac{1}{2}\sinh x - \frac{1}{2} x \sinh x - \frac{1}{2}x \cosh x,$

$u_2(x) = \frac{1}{4}\sinh x + \frac{1}{4} x \sinh x + \frac{1}{4}x^2 \sinh x + \frac{1}{4}x \cosh x + \frac{1}{4}x^2 \cosh x$

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$\varphi(x) = u(x) = u_0(x) + u_1(x) + \ldots\ldots\ldots\ldots\ldots$

$= \cosh x$

Example 3: Consider the following non-linear Volterra integral equation

$$y(x) = x - \int_0^x ((x-t)u(t))dt \; ,$$

the exact solution $\varphi(x) = \sin x$

First solve by Homotopy perturbation method:

We define

$F(u) = \varphi(x) - x$

$A(u) = \varphi(x) - x + \int_0^x ((x-t)u(t))dt$

and substituting F(u) and A(u) in (6) and equating the terms with identical power of p, we obtain

$P^0 : \varphi_0(x) = x$

$P^1 : \varphi_1(x) = -\int_0^x ((x-t)y_0(t))dt = -\int_0^x (x-t)t\,dt = -\frac{x^3}{6}$

$P^2 : \varphi_2(x) = -\int_0^x ((x-t)y_1(t))dt = -\int_0^x (x-t)\left(-\frac{t^3}{6}\right)dt = -\frac{x^5}{120}$

$P^3 : \varphi_3(x) = -\int_0^x ((x-t)y_2(t))dt = -\int_0^x (x-t)\left(-\frac{t^5}{120}\right)dt = -\frac{x^5}{5040}$

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$P^n : \varphi_n(x) = -\int_0^x (x-t)u_{n-1}(t)dt$

From equation (7), as follow

$\varphi(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots\ldots\ldots\ldots\ldots = \sin x$

Now solve by Variational Iteration Method:
The corresponding iterative relation (12) for this example can be constructed as:

$$\varphi_{n+1}(x) = \varphi_n(x) - \int_0^x \left(\varphi_n'(x) - 1 + \int_0^s \varphi_n(t)dt\right)dx$$

by taking $u_0(x) = x$,
we derive the following results:

$$u_1(x) = x - \frac{x^3}{6}$$

$$u_2(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$u_3(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$u_4(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\varphi(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots\ldots\ldots\ldots\ldots = \sin x$

Conclusion

In this work, homotopy perturbation method and variational iteration method have been successfully applied to find the solution of Volterra integral equations. The two methods can be concluded that the method is very powerful and efficient techniques in finding exact solutions or approximate solutions for wide classes of problems.In above examples, we have proved using both HPM and VIM has same solution in Volterra integral equations

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