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Stability analysis for a model of natural resources

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Abstract— **We study a two dimensional model for population and renewable resources. The model is given by a system of two differential equations depended on parameters. We analyze the nonlinear dynamics through bifurcation theory and numerical investigation. The system consists of various patterns depending on the parameters and initial conditions. Dynamics of the system is governed by equilibria relating to technical progress, culture and climate.**

Keywords- Equilibria, Bifurcations, Population, Renewable resource, Stability

I. INTRODUCTION

Understanding the resilience of the natural environment take an important role in determining the failure or success of societies in the sustainable exploitation of basic renewable resources.

A simple predator-prey model for renewable resource is proposed by Brander and Taylor [3]. A lots of other authors have developed this model with additional aspects such as institutions, property rights and technical progress (e.g. see [2], [4], [5], [7] and [8]). They show that the society always approaches a long-term steady state with positive stock of natural resource and positive population.

In this paper, we study a model for renewable resource and population proposed by D'Alessandro in [1]. The system is assumed to be homogenous. We pay more attention on the different features of different resources and find initial conditions of social-ecological system for sustainable development .

Our emphasis lies on obtaining a mathematical understanding of the dynamics and bifurcations of the model. We analyze the stability of equilibria and exhibit phase portraits for dynamics. Saddle-node and Hopf bifurcations are useful to explain the exchange of patterns. Moreover, a homoclinic bifurcation indicates the termination of periodic orbits. Numerical method is also used to study behaviour of the model. The numerical investigation is carried out by the software Mathematica and AUTO [5]. It is found that the proposed model reflects many stability properties.

II. THE STRUCTURE OF THE MODEL

A. *System of equations for the model*

The structure of the economics contains two components which exploit two different natural resources. A renewable resource (forest) and an inexhaustible resource (land) producing goods such as wood, corn, ... The survival of human community depend on these resources. Take into account the growth of the forest, it is assumed that the forest has a finite carrying capacity, that can not exceed a certain threshold, the forest is unable to recover. For the human population, it is assumed that it increases when the interest from land is over some average value and it decays when the interest is below this average value.

In accordance with the previous assumptions, one can formulate a two-dimensional system of nonlinear differential equations with the trees and human populations (see [1] for the details). The system has the following form:

$$\begin{cases} \dot{S} &= \left[\rho(S/k_1-1)(1-S/k_2)-\alpha\beta L\right]S \\ \dot{L} &= \gamma\left[\lambda(1-\beta)^{\delta}L^{\delta-1}+p\alpha\beta S-\sigma\right]L \end{cases} \tag{1}$$

where S stands for tree population in the forest (resource stock), L regards human population and parameters given by the following table.

TABLE I. TABLE OF PARAMETERS GIVEN IN EQUATIONS (1)

Parameters	Meaning	Ranges
α	The technical parameter	$\alpha > 0$
β	The quota of income spent on wood	$\beta \in (0,1)$
δ	The technical parameter	$\delta \in (0, 1)$
γ	The caloric unit of corn	$\gamma > 0$
k_1	The threshold level	$k_1 > 0$
k_2	Carrying capacity	$k_2 > 0$
λ	Index of land fertility	$\lambda > 0$
ρ	The intrinsic regeneration rate	$\rho > 0$
p	$p=\phi/\gamma$, where ϕ is the caloric unit of wood	$p > 0$
σ	The per capita natural caloric level	$\sigma > 0$

The parameters α, δ and λ are related to technical progress; the parameters β and σ present cultural changes; and the parameters k_1, k_2, ρ stand for climate changes.

We establish the invariant set of the system (1) that is the first quadrant:

$$D = \{(S(t), L(t)) : S(t) \geq 0, L(t) > 0 \ \ \forall t \geq 0\}$$

B. Equilibria

To find equilibria, we set the right-hand side of the system (1) equal to zero. There are at least four equilibria:

$$O(0,0),\ E_1(k_1, 0),\ E_2(k_2, 0),\ E_3(0, \bar{L})$$

where $\bar{L} = \left(\frac{\lambda(1-\beta)^{\delta}}{\sigma}\right)^{1/(1-\delta)}$.

The equilibria O, E_1 and E_2 are irrelevant for the aim of our study because the system can not reach $L = 0$ when the initial value of population is positive, i.e. when $L(0) > 0$.

Moreover, we also have other equilibria. They are positive solutions of the following system

$$\begin{cases} \rho(S/k_1 - 1)(1 - S/k_2) - \alpha\beta L & = & 0 \\ \lambda(1-\beta)^{\delta} L^{\delta-1} + p\alpha\beta S - \sigma & = & 0 \end{cases}. \qquad (2)$$

The local stability for equilibria is determined by the Jacobian matrix of the system (1), which is

$$J = \begin{pmatrix} -\frac{3\rho}{k_1 k_2} S^2 + 2\rho\left(\frac{1}{k_1} + \frac{1}{k_2}\right) S - \rho - \alpha\beta L & -\alpha\beta S \\ \gamma p\alpha\beta L & \gamma\left[\delta\lambda(1-\beta)^{\delta} L^{\delta-1} + p\alpha\beta S - \sigma\right] \end{pmatrix}$$

Eigenvalues for each equilibrium is obtained by solving the characteristic equation

$$\det(J - \lambda I) = 0.$$

O(0, 0) has negative eigenvalues $\lambda_1 = -\rho$ and $\lambda_2 = -\gamma\sigma$. Therefore, O is always a stable equilibrium.

$E_1(k_1, 0)$ has eigenvalues $\lambda_1 = (1 - \frac{k_1}{k_2})\rho$, $\lambda_2 = \gamma\alpha\beta p k_1 - \gamma\sigma$.

$E_1(k_2, 0)$ has eigenvalues $\lambda_1 = (1 - \frac{k_2}{k_1})\rho$, $\lambda_2 = \gamma\alpha\beta p k_2 - \gamma\sigma$.

$E_3(0, \bar{L})$ has eigenvalues

$$\lambda_1 = -r - \alpha\beta\left(\lambda(1-\beta)^{\delta}/\sigma\right)^{1/(1-\delta)} < 0 \ , \ \ \lambda_2 = (\delta - 1)\gamma\sigma < 0.$$

This implies E_3 is always stable.

C. Analysis of the model

The dynamics of the model is controlled by the equilibrium E_3 and interior equilibria which corresponding to the positive solutions of the system (2). To simplify the analysis of stability, we study in two main cases.

Case I: The system has no interior solutions

In this case, the equilibrium $E_3 = (0, \bar{L})$ is only one stable equilibrium and it is stable node. The dynamical system can be reached this point, and therefore, it is global stable. The resource stock will be exhausted in the long term and the population will tend to the value $\bar{L} = \left(\lambda(1-\beta)^{\delta}/\lambda\right)^{1/(1-\delta)}$. This mean that the regeneration rate of the renewable resource is very low with respect to the growth rate of the human population.

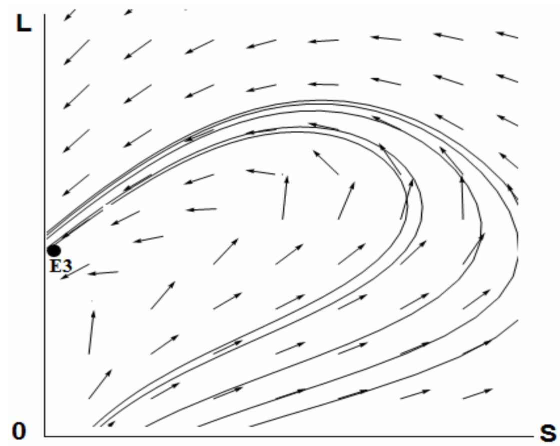

Figure 1. Phase trajectories near the equilibrium E_3 for the dynamics behaviour of the model for α = 0.001, β = 0.3, δ = 0.7, γ = 0.1, λ = 19.75, k_1 = 700, k_2 = 12.000, p = 3, ρ = 0.025, σ = 1.8 (Case I).

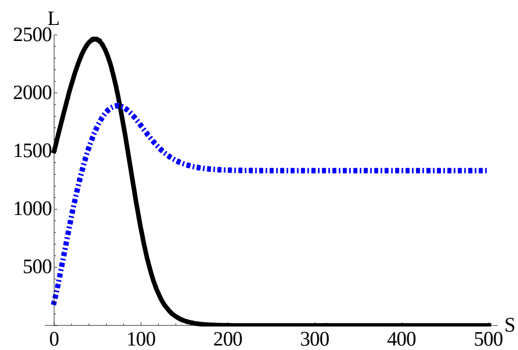

Figure 2. Time series of S and L in case I.

Fig. 1 shows a phase portrait of the system. Solution trajectories tend to the equilibrium E_3 where the resource stock is zero. Fig.2 presents time series of solutions. Solid line is the line of resource stock S and dash line is the line of human population L. When time is very large, values of S almost vanish while values of tend to a certain value $\bar{L} = 1331$.

Case II: The system has interior solutions

When there exists interior equilibria of the system (1) that are the positive solutions of equation (2). The behaviour of the dynamics is more complicated. The positive solution is locally stable. We note that the equilibrium E_3 is always stable. There are two scenarios depending on the stability of the interior equilibrium.

In the first scenario, two interiors are unstable. The equilibrium E_3 is global stable. In this case, solutions of the system approach asymptotically to E_3 or converge in a periodic orbit around the upper interior equilibrium (in compare with L). We note that periodic orbits are created by Hofp bifurcation.

The second scenario is the case in which the upper interior equilibrium is stable. For this case, the initial conditions will determine either E_3 or the interior equilibrium control the system. A small change in parameters relating to technology, culture or climate makes a strong divergence in the long-term of solutions.

Fig. 3 and Fig. 4 indicate the dependence of solutions on initial conditions. With the same values of parameters, Fig. 3 shows the solutions tending to the interior equilibrium S = 5550, L = 2820 when start from the initial condition S(0) = 1670.5, L(0) = 5000, while Fig. 4 presents the solutions approaching to the equilibrium $E_3(0, 505)$ when begin from the initial condition S(0) = 200, L(0) = 3500. In two figures, the solid line is for S and the dash line is for L.

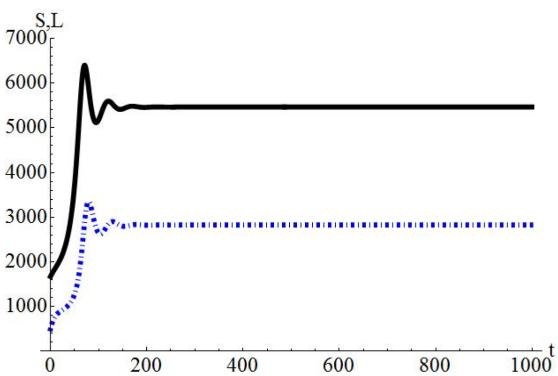

Figure 3. Time series of S and L in case II for $\alpha = 0.0001094$, $\beta = 0.3$, $\delta = 0.7$, $\gamma = 0.1$, $\lambda = 12$, $k_1 = 700$, $k_2 = 12.000$, $p = 3$, $\rho = 0.025$, $\sigma = 1.4$ and an initial condition S(0) = 1670.5, L(0) = 5000.

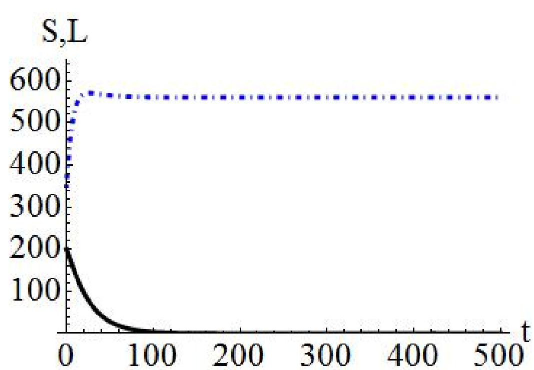

Figure 4. Time series of S and L in case II for $\alpha = 0.0001094$, $\beta = 0.3$, $\delta = 0.7$, $\gamma = 0.1$, $\lambda = 12$, $k_1 = 700$, $k_2 = 12.000$, $p = 3$, $\rho = 0.025$, $\sigma = 1.4$ and an initial condition S(0) = 1670.5, L(0) = 5000.

There exists a saddle-node bifiurcation occur in the system, where a interior stable equilibrium changes its stability. A pair of equilibria is created. When the parameters vary, one unstable equilibrium under goes a Hofp bifurcation where the equilibrium loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. Arising from this bifurcation periodic orbits are branched from the equilibrium. It implies the existence of oscillation solutions of the system. This means oscillations of forest and population occur in the model. The periodic orbits expand with large periods and end at a homoclinic cycle.

This case contain a sustainable development of the social-ecilogycal system which corresponding homoclinic cycle or convergence of solutions of the model to interior equilibrium.

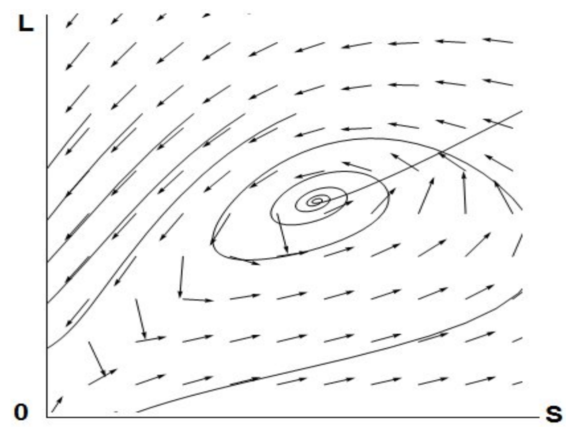

Figure 5. Phase trajectories near the interior equilibrium for the dynamics behaviour of the model for $\alpha = 0.001$, $\beta = 0.3$, $\delta = 0.7$, $\gamma = 0.1$, $\lambda = 18.5$, $k_1 = 700$, $k_2 = 12.000$, $p = 3$, $\rho = 0.025$, $\sigma = 1.75$ (Case II).

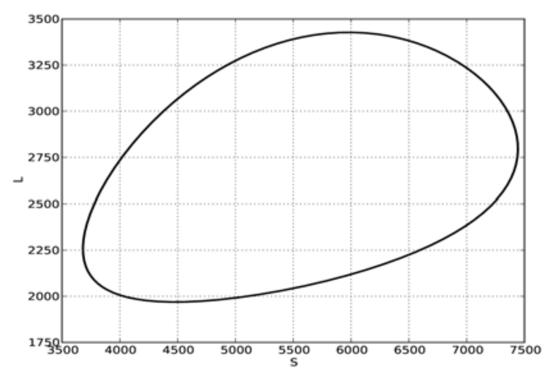

Figure 6. A oscillation solution of the system for $\alpha = 0.0001104$, $\beta = 0.3$, $\delta = 0.7$, $\gamma = 0.1$, $\lambda = 12$, $k_1 = 700$, $k_2 = 12.000$, $p = 3$, $\rho = 0.025$, $\sigma = 1.4$.

III. BIFURCATION INVESTIGATION

In this section we carry out a numerical investigation to detect bifurcations of the system (1). We use the software Mathematica for computing, and the software package AUTO [5] for detecting bifurcations .

Case I.

In this case the system has four equilibria. Interior equilibria, corresponding to positive solutions, are absent. E_3 is an unique stable equilibrium and it is global stable. The dynamical behaviour is very simple. Solutions of the system approach to E_3 as time tend to infinity. Fig. 7 shows bifurcation diagram of the equilibrium E_3 for $\alpha = 0.0001$, $\beta = 0.3$, $\gamma = 0.1$, $\lambda = 19.85$, $k_1 = 700$, $k_2 = 12000$, $p = 3$, $\rho = 0.025$ and $\sigma = 1.8$. Using the software package AUTO, we continue from the interior equlirium $E_3(0, 1331.74)$ and let δ varies, then we get the curve for E_3 in the plane (δ, S) . There is no bifurcation in this case. It is found that the dynamics of the model is determined by stability of E_3. Values of L for E_3 is increase when δ becomes large.

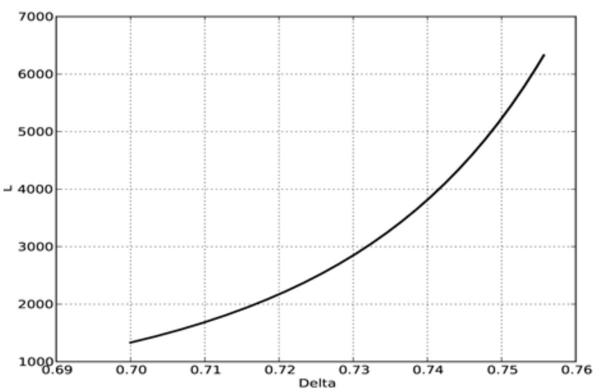

Figure 7. Bifurcation diagram of the equilibrium E_3.

Case II.

For this case, the system has four equilibria O, E_1, E_2, E_3 and two interior equilibria. As mentioned before, depending on the initial condition ether E_1 or interior equilibria will control the dynamics of the system. We concentrate our study in the interior equilibria because they make the system more complicated.

Fixing the following parameters $\beta = 0.3$, $\delta = 0.7$, $\gamma = 0.1$, $\lambda = 12$, $k_1 = 700$, $k_2 = 12000$, $p = 3$, $\rho = 0.025$ and $\sigma = 1.4$ and let α varies. Continue from an interior stable equilibrium (S, L) with S = 1220.66, L = 695.979, AUTO detects a saddle-node bifurcation LP occurs at value $\alpha = 0.0001351$. A pair of interior equilibria appear and they are unstable. Path following from LP, AUTO found a Hopf bifurcation H for $\alpha = 0.000108$. Numerical evidence shows that this is generalized Hopf bifurcation. Periodic orbits, corresponding to oscillation solutions is created. These orbits are stable. They attract points of the system. In order to find the end of limit cycles we continue H in periods, AUTO found that periodic orbits terminate at a homoclinic cycle which can be seen as a cycle with infinity period.

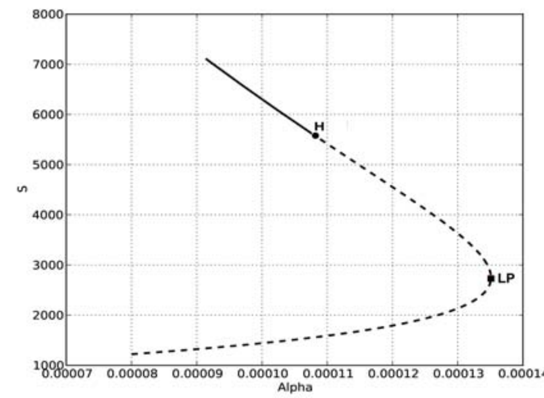

Figure 8. Bifurcation diagram of interior equilibrium

Fig. 8 shows bifurcation diagram of interior equilibria in the plane (α, S) where LP is point for saddle-node bifurcation and H is for Hopf bifurcation. The solid line is for stable equilibria and the dash line is for unstable equilibria . Fig. 9 presents the development of periodic orbits in periods. These orbits end at a homoclinic cycle.

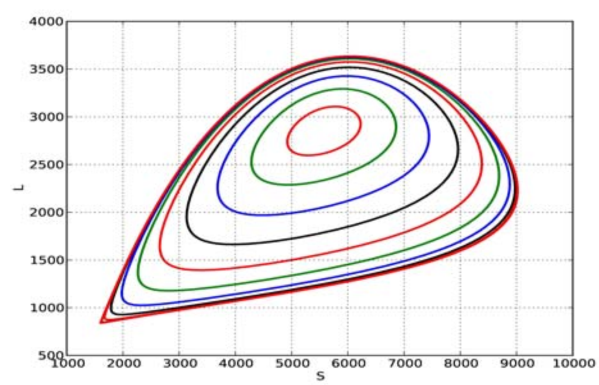

Figure 9. Periodic orbits and homoclinic cycle

We carried out several investigations for bifurcation diagrams with all available parameters, and we always obtain the same mechanism.

IV. CONCLUSIONS

In this paper, a model of social-ecological system with the dynamic interaction between renewable resource and population is studied. Dynamical behaviour of this system is investigated from equilibrium points. It has been shown that the positive solutions possess saddle-node and Hopf bifurcations, as one parameter is varied, the dynamics of the system near to this solution changes the stability. Both analytically and numerically, simulation shows that in the parameter space, the model sensitively depends on the parameter values and initial conditions. A suitable initial condition of social-ecological system can make a sustainable exploitation.

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REFERENCES

[1] S. D'Alessandro, "Non-linear dynamics of population and natural resources: The emergence of different patterns of development", *Ecological Econimics*, vol. 62, pp. 473-481, May 2007.

[2] J. Anderie, "Culture and Human Agro-ecosystem Dynamics: the Tsembaga of New Guinea", *Journal of Theoretical Biology*, 192 (4), pp. 515-530, 1998.

[3] J. A. Brander and M. S. Taylor, "The Simple, Economics of Easter Island: a Ricardo-Malthus Model of Renewable Resource Use ", *American Economic Review*, 88 (1), pp. 515-530, 1998.

[4] R. Dalton and R. M. Coats, "Could nstitution reform savea Easter Island?", *Journal of evolutionary Economics*, 10, pp. 489-505, 2000.

[5] E. J. Doedel, R.C. Paffenroth, A.R. Champneys, T.F. Fairgrieve, Y.A. Kuznetsov, B. Sandstede, X. Wang, *AUTO 2000*: *Continuation and Bifurcation Software for Ordinary Differential Equations,*

http:// sourceforge.net/projects/auto2000/.

[6] J. . Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* New York: Springer-Verlag, 1983.

[7] C. V. Pezzey and J. M. Anderies, "The effect of subsistence on collapse and institutional adaptation in Population-Resource Societies", *Journal of Sevelopment Economics*, 72, pp. 299-320, 2003.

[8] R. Reuveny and C. S. Decker, "Easeter Island: Historical anecdote or warning for the future", *Ecologycal Economics*, 35, pp. 271-287, 2000.

[9] X. Q. Shi, "Dynamic analysis and assessment for sustainable development", *Journal of Environmental Sciences*, 14, (1), pp 88 - 94, 2002.

[10] B. Walker, C.S. Holling, S. R. carpenter and A. Kinzig, "Resilience, adaptability and trnsformability in social-ecology systems ", *Ecology and Society*, 9 (2): art.5, 2004.