Circular Model Induced by Inverse Stereographic Projection On Extreme-Value Distribution

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Abstract—Minh and Farnum proposed a new method of generating probability distributions on real line induced by Stereographic Projection. Toshihiro Abe also worked on Symmetric Circular Models. Motivated by these works we made an attempt to construct Asymmetric Circular Model induced by Inverse Stereographic Projection and is coined as Stereographic Extreme – value distribution. We derive the characteristic function of this circular model induced by inverse stereographic projection and also study the Characteristics of the newly generated Asymmetric Circular Model. Goodness of fit is verified for the data set of Movements of Turtles Orientations of 76 turtles after laying eggs which contains angular data and role of concentration parameter is presented.

Keywords: Characteristic function; Circular models; Inverse Stereographic Projection; Measurable function; trigonometric moments; unimodal.

I- Introduction

By wrapping some life testing models on a unit circle, Dattatreya Rao et al [2] derived new circular models. It is proposed that one way of constructing Circular Models is to apply Stereographic Projection on linear models. Minh & Farnum [9] proposed a new method of generating probability distributions by applying Inverse Stereographic Projection, which maps every point on real line onto the point on Unit circle. Dattatreya Rao et al [3] applied stereographic projection on Cardioid model to generate Cauchy Type models and also presented a differential approach to circular models. Various types of constructing circular models are discussed in Jammalamadaka & Sen Gupta [6] and Girija [5]. On the lines of Minh and Farnum and Toshihiro Abe et al [9, 12], we made an attempt to construct Stereographic Extreme – value Model.

II.Methodology of Inverse Stereographic Projection

Inverse Stereographic Projection is defined by a one to one mapping given by $T(\theta) = x = u + v\tan\left(\frac{\theta}{2}\right)$, where $x \in (-\infty, \infty)$, $\theta \in [-\pi, \pi)$, $u \in \mathbb{R}$, and $v > 0$,then by Minh and Farnum [9] $T^{-1}(x) = \theta = 2\tan^{-1}\left\{\frac{(x-u)}{v}\right\}$ is a random point on the Unit circle. Suppose x is randomly chosen on the interval $(-\infty, \infty)$. Let $F(x)$ and $f(x)$ denote the Cumulative distribution and probability density functions of the random variable X on the real line respectively. Also $G(\theta)$ and $g(\theta)$ denote the Cumulative distribution and probability density functions of this random point θ on the unit circle respectively. Then $G(\theta)$ and $g(\theta)$ can be written in terms of $F(x)$ and $f(x)$ using the following Theorem 2.1 as stated below.

Theorem 2.1: For $v > 0$,

i) $G(\theta) = F\left(u + v\tan\left(\frac{\theta}{2}\right)\right) = F(x(\theta))$

ii) $$g(\theta) = v\left(\frac{1+\tan^2\left(\frac{\theta}{2}\right)}{2}\right) f\left(u + v\tan\left(\frac{\theta}{2}\right)\right)$$

$$= v\left(\frac{1+\left(\frac{x(\theta)-u}{v}\right)^2}{2}\right) f\left(\theta(x)\right)$$

III. Stereographic Extreme-Value Distribution

A. Definition

A continuous random variable X on the real line is said to have Extreme-Value Distribution with location parameter γ and scale parameter $\lambda > 0$, if the probability density function and cumulative distribution function of X are given by

$$f(x) = \frac{1}{\lambda}\exp\left(\frac{-(x-\gamma)}{\lambda}\right)\exp\left(-\exp\left(\frac{-(x-\gamma)}{\lambda}\right)\right),$$

where $\lambda > 0$, and $\gamma,\ x \in \mathbb{R}$ and

$$F(x) = \exp\left(-\exp\left(\frac{-(x-\gamma)}{\lambda}\right)\right) \text{ respectively.}$$

B. Definition

A random variable X_S on unit circle is said to have Stereographic Extreme-Value Distribution with location parameter μ scale parameter $\sigma > 0$ denoted by SEV$(\mu,\ \sigma)$

Then by applying Inverse Stereographic Projection defined by a one to one mapping

$$x = u + v\tan\left(\frac{\theta}{2}\right),\ v > 0, -\pi \le \theta < \pi,$$

Which leads to a Circular Stereographic Extreme-Value Distribution on unit circle, whose probability density function and cumulative distribution function are given by

$$g(\theta) = \frac{1}{2\sigma}\sec^2\left(\frac{\theta}{2}\right)\exp\left(-\frac{\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)$$

$$\exp\left\{-\exp\left(-\frac{\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right\},$$

Where, $\mu = \frac{\gamma}{v},\ \sigma = \frac{\lambda}{v} > 0$

$$G(\theta) = \exp\left\{-\exp\left(-\frac{\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right\},$$

where $\sigma > 0,\ -\pi \le \theta < \pi,$ respectively.

Clearly:

1. $g(\theta) \ge 0,\ \forall\ \theta \in [-\pi, \pi)$
2. $g(\theta + 2\pi k) = g(\theta),\ k \in \mathbb{Z}$
3. $\int_{-\pi}^{\pi} g(\theta)\,d\theta = 1$

Theorem 3.1: Stereographic Extreme –Value distribution is **unimodal** if $\sigma < \frac{1}{\sqrt{2}}$ and **bimodal** if $\sigma > \frac{1}{\sqrt{2}}$

Proof: The probability density function of Stereographic Extreme-Value Distribution is

$$g(\theta) = \frac{1}{2\sigma}\sec^2\left(\frac{\theta}{2}\right)\exp\left(-\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)\exp\left\{-\exp\left(-\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)\right\}$$

where $\sigma > 0, -\pi \le \theta < \pi$

Differentiating $g(\theta)$ with respect to θ, we get

$$g^{'}\left(\theta\right)=\frac{1}{2\sigma}\sec^{2}\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)}e^{-e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)}}$$

$$.\left[\tan\left(\frac{\theta}{2}\right)-\frac{1}{2\sigma}\sec^{2}\left(\frac{\theta}{2}\right)+\frac{1}{2\sigma}\sec^{2}\left(\frac{\theta}{2}\right)e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)}\right]$$

$\theta=0,\ 2\pi,\ 4\pi,...,$ are the stationary points.

$\theta=0$ is the only stationary point which lies in the domain of $g\left(\theta\right)$

At $\theta=0$

$$g^{''}\left(0\right)=\frac{2\sigma^{2}-1}{8\sigma^{3}e}$$

$g\left(\theta\right)$ has maximum value at $\theta=0$, if and only if,

$$g^{''}\left(0\right)<0$$

$$\Leftrightarrow\left[\frac{2\sigma^{2}-1}{8\sigma^{3}e}\right]<0$$

$$\Leftrightarrow 2\sigma^{2}-1<0$$

$$\Leftrightarrow \sigma<\frac{1}{\sqrt{2}}$$

$g\left(\theta\right)$ has minimum value at $\theta=0$, if and only if,

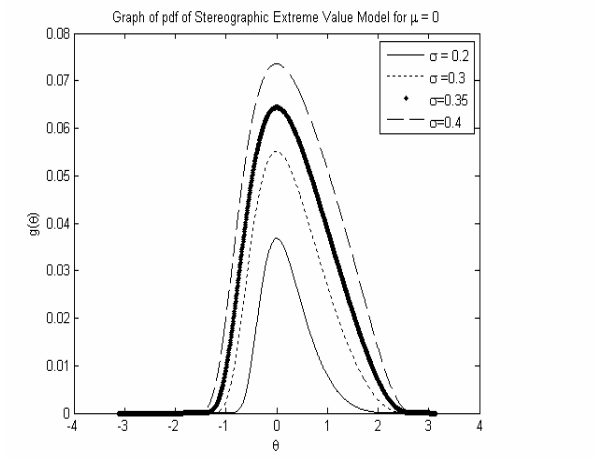
$$g^{''}\left(0\right)>0$$

$$\Leftrightarrow\left[\frac{2\sigma^{2}-1}{8\sigma^{3}e}\right]>0$$

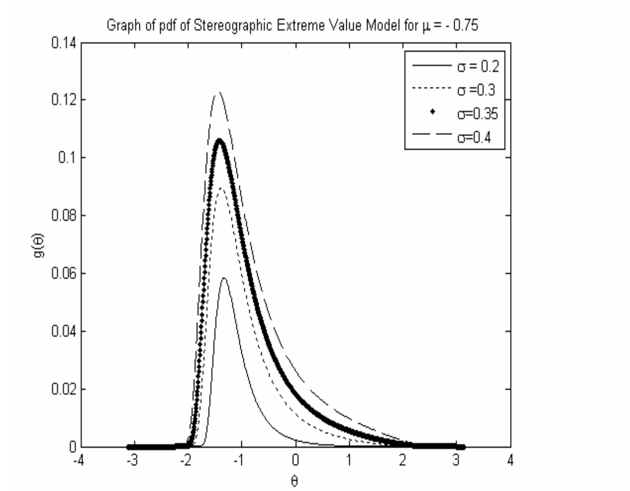
$$\Leftrightarrow \sigma>\frac{1}{\sqrt{2}}$$

Hence Stereographic Extreme-Value distribution is unimodal if $\sigma<\frac{1}{\sqrt{2}}$ and bimodal if $\sigma>\frac{1}{\sqrt{2}}$

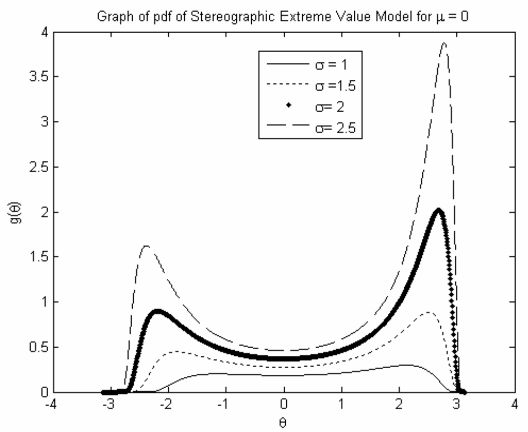
C. Graphs of probability density function and cumulative distribution function of Stereographic Extreme-Value Distribution for various values of σ and μ are presented here.



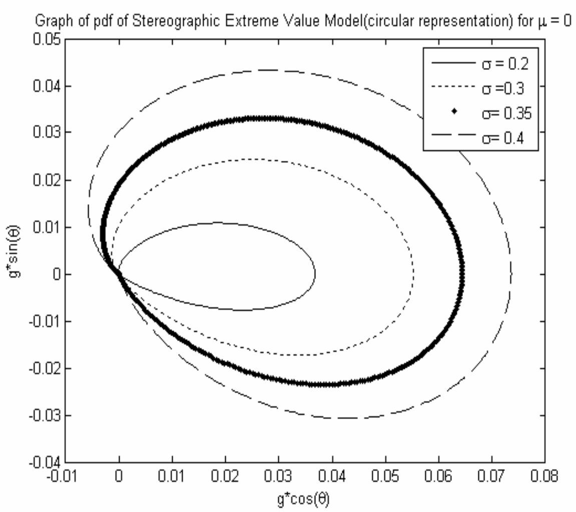
(Unimodal) Fig 1



(UNIMODAL) FIG-2.



(BIMODAL) FIG3.



(Circular Represenation) Fig-4

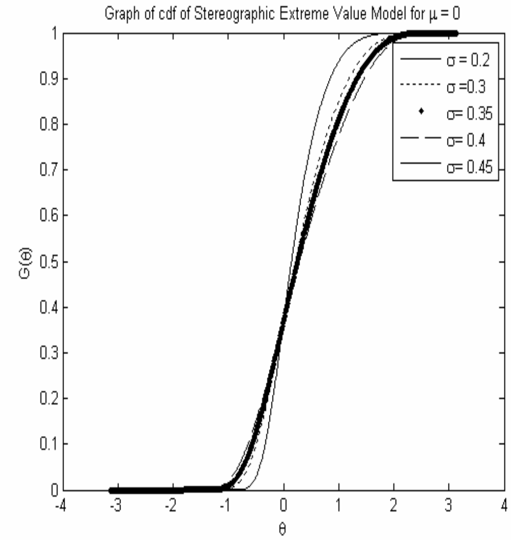


Fig 5

D. Role of parameter σ :

The larger the value of σ , the larger will be the ratio of $g(\mu)$ to $g(\mu \pm \pi)$ indicating higher concentration towards the polar direction μ (here $\mu = 0$).Thus, $\frac{1}{\sigma}$ is a parameter which measures the concentration towards the mean direction. It is also clear from the graph given below.

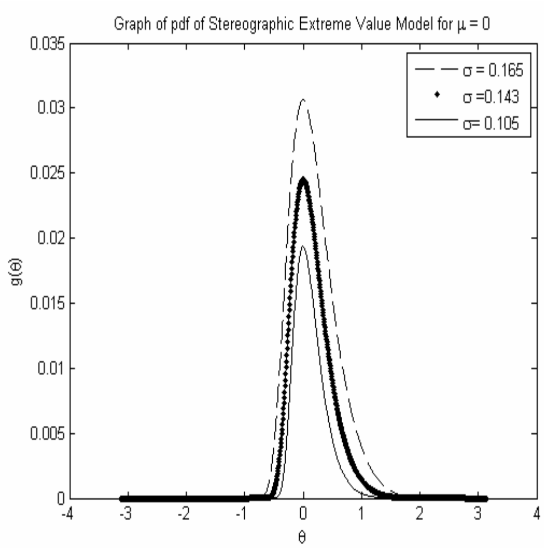


Fig 6

IV. Characteristic function of Stereographic Model

The Characteristic function of a Circular model with probability density function $g(\theta)$ is defined as $\varphi_p(\theta) = \int_0^{2\pi} e^{ip\theta} g(\theta)\, d\theta$, $p \in \mathbb{R}$. Ramabhadra Sarma et al [10, 11] derived the characteristic functions of some new wrapped models. The characteristic function of a Stereographic Circular model can be obtained in terms of respective linear model. Lukacs [7] proved the following theorem related to the Characteristic function of linear model which is applied here in the case of Stereographic Circular Models.

Theorem 4.1: Let X be a random variable with distribution function $F(x)$ and suppose that $S(x)$ is a finite, single-valued and B-measurable function of *x*. The Characteristic function of $f_Y(t)$ of the random variable $Y = S(x)$ is then given by $f_Y(t) = E\left(e^{itY}\right) = E\left(e^{itS(X)}\right) = \int_{-\infty}^{\infty} e^{itS(X)} dF(x)$.

By applying the above theorem we derive the Characteristic function of a Stereographic Circular model.

Theorem4.2 : If $G(\theta)$ and $g(\theta)$ are the cdf and pdf of the Stereographic Circular model and $F(x)$ and $f(x)$ are cdf and pdf of the respective linear model, then Characteristic function of Stereographic Model is $\varphi_{X_S}(p) = \varphi_{2\tan^{-1}\left(\frac{x}{v}\right)}(p),\ p \in \mathbb{R}$

Proof:

$$\varphi_{X_S}(p) = \int_{-\pi}^{\pi} e^{ip\theta} d\left(G(\theta)\right) \quad ,\ p \in \mathbb{R}$$

$$= \int_{-\pi}^{\pi} e^{ip\theta} d\left(F\left(v \tan \frac{\theta}{2} \right) \right)$$

$$= \int_{-\infty}^{\infty} e^{ip\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)} f(x)\, dx, \quad \text{taking } x = v \tan\left(\frac{\theta}{2}\right)$$

$$= \varphi_{2\tan^{-1}\left(\frac{x}{v}\right)}(p)$$

As the integral cannot be obtained analytically, MATLAB Techniques are applied for the evaluation of the values of the characteristic function. The graphs for real and imaginary parts of the characteristic function are plotted here. Numerical integration of Gauss – Laguerre is used for the computation of the values of the characteristic function of the Stereographic circular model.

A. The Characteristic function of Stereograph Extreme-Value Model is

$$\Phi_{X_S}(p) = \int_{-\pi}^{\pi} e^{ip\theta} g(\theta)\, d\theta$$

$$= \int_{-\infty}^{\infty} e^{ip\left(2\tan^{-1}\left(\frac{x}{v}\right)\right)} \frac{1}{\sigma} e^{-\left(\frac{x}{\sigma}\right)} e^{-e^{-\left(\frac{x}{\sigma}\right)}} dx$$

Mardia and Jupp [8] gave expressions of mean direction, resultant length, circular variance, circular standard deviation, central trigonometric moments, skewness and kurtosis for circular distributions. These characteristics for the Stereographic Extreme-Value model are also based on their respective trigonometric moments. These can be expressed in terms of trigonometric moments α_p and β_p and are presented here.

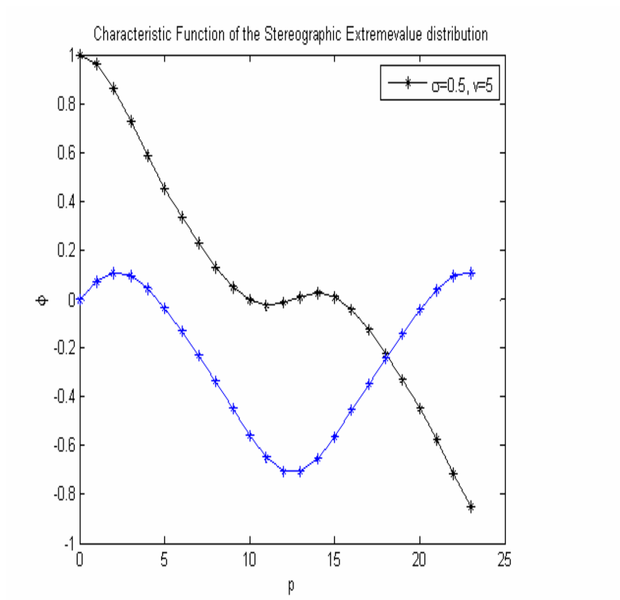


Fig 7

Table 1 Characteristics of Stereographic Extreme -Value distribution

Stereographic Extreme-Value Distribution	σ = 0.1	σ = 0.15	σ = 0.2	σ = 0.53	σ = 0.4
Mean Direction μ	0.0734	0.1625	0.2105	0.2662	0.3215
Trigonometric Moments α_1 α_2 β_1 β_2	0.9622 0.8615 0.0707 0.1085	0.9250 0.7454 0.1517 0.2157	0.8771 0.6099 0.1874 0.2267	0.7813 0.3849 0.2130 0.1556	0.6857 0.2143 0.2283 0.0790
Resultant length ρ	0.9648 0.8683	0.9374 0.7760	0.8969 0.6506	0.8098 0.4152	0.7227 0.2284
Variance V_o	0.0352	0.0626	0.1031	0.1902	0.2773
Central Trigonometric Moments α_1^* α_2^* β_1^* β_2^*	0.9648 0.8681 0.0000 -0.0186	0.9374 0.7753 0.0000 -0.0337	0.8969 0.6492 0.0000 -0.0424	0.8098 0.4106 0.0000 -0.0613	0.7227 0.2189 0.0000 -0.0653
Skewness γ_1^o	-2.8268	-2.1469	-1.2791	-0.7390	-0.4472
Kurtosis γ_2^o	1.2324	0.8277	0.2034	-0.5344	-0.7012
Circular Standard Deviation σ_o	0.2676 0.5314	0.3597 0.7122	0.4665 0.9272	0.6496 1.3259	0.8059 1.7185

V. Goodness of Fit for Live Data

For the purpose of verifying goodness of fit the following **"Live"** data set is considered. **Data Set:** Movements of Turtles Orientations of 76 turtles after laying eggs [Rao Jammalamadaka and Sen Gupta [6] p.5]

8	9	13	13	14	18	22	27	30	34
38	38	40	44	45	47	48	48	48	48
50	53	56	57	58	58	61	63	64	64
64	65	65	68	70	73	78	78	78	83
83	88	88	88	90	92	92	93	95	96
98	100	103	106	113	118	138	153	153	155
204	215	223	226	237	238	243	244	250	251
257	268	285	319	343	350				

The above data set is used to verify goodness of fit of Stereographic Extreme- value model.

A. Methodology:
(Tracing density function for the estimated parameters)

Considering the data, using the formulae in Rao Jammalamadaka and Sen Gupta [6] the mean direction (μ) and the concentration parameter (σ) are estimated to be $\hat{\mu} = 1.12$ and $\hat{\sigma} = 0.4971$.

Using the above $\hat{\mu}$ and $\hat{\sigma}$ values, the graph of the density function is drawn, plotting the data points on it and are presented in Fig **7** for Stereographic version of Extreme – value model.

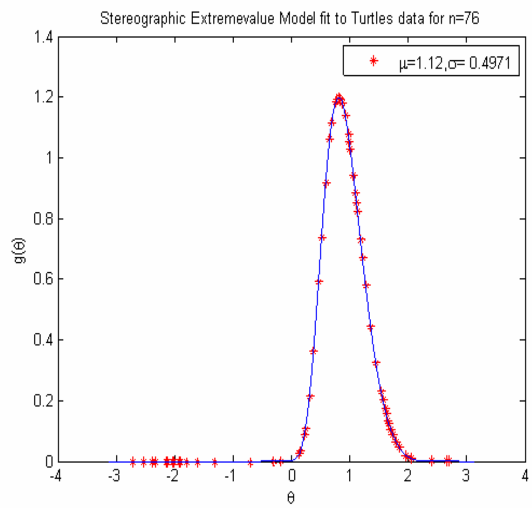


Fig 8

From the above figure it is clear that the Stereographic Extreme – value model is a good fit as **"almost all points lie on the curve".**

B.Methodology
(Using Tests of Uniformity)

Substituting data set in the cdf of the said model, corresponding uniform $[-\pi,\pi)$ variates denoted by $\theta_1,\theta_2,...,\theta_n$ are obtained. Using these θ_i's, the tests statistics of Rayleigh Test, Kuiper's Test, Watson's U^2- Test, Hodges –Ajne Test, Range Test, Rao's Equal Spacing Test and Ajne Test are computed and are tabulated in Table 2.

Table 2 Statistic Values of Various Test Procedures

Tests	Test Statistic for Sample Size (n) = 76
Rayleigh Test	16.8967
Kuiper's Test	4.4247
Watson's U^2 - Test	0.0011
Hodges –Ajne Test	39.000
Range Test	**5.7766**
Rao's Equal Spacing Test	8.5948
Ajne Test	4.7433

The cut of points for the above sample sizes are taken from Devaraaj [4].

LOS / Tests	1%	5%	10%
Rayleigh Test	0.0130 - 11.605	0.0525 - 7.6470	0.0930 - 6.0220
Kuiper's Test	0.6900 - 2.0576	0.7806 - 1.8307	0.8391 - 1.7297
Watson's U^2- Test	0.0170 - 0.3131	0.0234 - 0.2246	0.0270 - 0.1871
Hodges –Ajne Test	10.000 - 24.000	12.000 – 23.000	13.000 – 22.000
Range Test	**5.5090 - 6.0341**	**5.6467 - 6.0132**	**5.6987 - 6.0006**
Rao's Equal Spacing Test	1.8170 - 2.8022	1.9424 - 2.6573	2.0011 - 2.6138
Ajne Test	0.0270 - 1.0430	0.0383 - 0.7951	0.0484 - 0.6521

Remark:

Stereographic Extreme Value model appears to be good fit at 1%, 5% and 10% based on Range test.

REFERENCES

[1] Abramowitz, M. & Stegun, I.A. (1965), Handbook of Mathematical Functions, *Dover, New York.*

[2] Dattatreya Rao, A.V., Ramabhadra Sarma, I. and Girija, S.V.S. (2007), On Wrapped Version of Some Life Testing Models, *Comm Statist,- Theor.Meth.* , 36, issue # 11, pp.2027-2035.

[3] Dattatreya Rao, A.V., Girija, S.V.S., Phani, Y. (2011), Differential Approach to Cardioid Distribution, Computer Engineering and Intelligent Systems, Vol 2, No.8, pp. 1-6.

[4] Devaraaj V. J.(2012), Some Contributions to Circular Statistics, Thesis submitted to Acharya Nagarjuna University for the award of Ph. D.

[5] Girija, S.V.S., (2010), Construction of New Circular Models, VDM – VERLAG, Germany.

[6] Jammalamadaka S. Rao and Sen Gupta, A. (2001), Topics in Circular Statistics, *World Scientific Press, Singapore.*

[7] Lukacs, E. and Laha, R.G. (1970), Applications of Characteristic Functions, *Charles Griffin and Company Limited, London, second edition.*

[8] Mardia, K.V. and Jupp, P.E. (2000), Directional Statistics, *John Wiley, Chichester.*

[9] Minh, Do Le and Farnum, Nicholas R. (2003), Using Bilinear Transformations to Induce Probability Distributions, *Communication in Statistics – Theory and Methods*, 32, 1, pp. 1 – 9.

[10] Ramabhadrasarma, I. A.V.Dattatreya Rao and S.V.S.Girija (2009), On Characteristic Functions of Wrapped Half Logistic and Binormal Distributions, *International Journal of Statistics and Systems*, Volume 4 Number 1, pp. 33–45.

[11] Ramabhadrasarma, I. A.V.Dattatreya Rao and S.V.S.Girija (2011), On Characteristic Functions of Wrapped Lognormal and Weibull Distributions, *Journal of Statistical Computation and Simulation*, Vol. 81, No. 5, 579–589.

[12] Toshihiro Abe, Kunio Shimizu and Arthur Pewsey, (2010), Symmetric Unimodal Models for Directional Data Motivated by Inverse Stereographic Projection, *J. Japan Statist. Soc.*, Vol. 40 (No. 1), pp. 45-61.