# THE Z-TRANSFORM OF THE LINEAR CAR-FOLLOWING MODEL

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***Abstract* - The Z-transform is used to analyze the differential equation for the linear car-following model. The car-following model has been developed to describe the dynamics of a platoon of vehicles initially travelling at uniform speed and spacing when the lead vehicle introduces a perturbation by decelerating or accelerating. The shifting theorem and the convolution theorem for Z-transform are used to model fluctuations in velocities and spacing of following vehicles as a function of deceleration or acceleration of the lead car when initial spacing and velocities are known. Fourier transform, the Laplace transform and differential operator methods are also discussed. The car-following model with delay reaction time is treated.**

***Key Words*- linear car-following; vehicle platoons; vehicle spacing; z-transform; shifting theorem; convolution theorem.**

## I. INTRODUCTION

The car-following model has been developed to describe the dynamics of a platoon of vehicles initially travelling at uniform speed and spacing when the lead vehicle introduces a perturbation by braking or accelerating. The linear car-following model was developed by Chandler, et al.[1]and Herman et al.[5]. Assume that $x_n(t)$ and $v_n(t)$ are the position and velocity functions of the $n^{th}$ car at the time $t$. $x_0(t)$ and $v_0(t)$ are the position and velocity functions of the lead vehicle. Chandler, Herman and Montroll [1] assume that the acceleration of the $n^{th}$ car with time delay $T$ is proportional to the relative velocities of the $n^{th}$ car and n-$1^{th}$ car at time $t$.

$$x_n^{''}(t+T) = -\lambda\left[x_n^{'}(t) - x_{n-1}^{'}(t)\right] \qquad (1)$$

The constant is called the sensitivity constant. If the delay time T is small, as should be the case in driving, we obtain the following second orderdifferential equation:

$$x_n^{''}(t) = -\lambda\left[x_n^{'}(t) - x_{n-1}^{'}(t)\right] \qquad (2)$$

From eq. (2), the corresponding equation for velocities is:

$$v_n^{'}(t) = -\lambda\left[v_n(t) - v_{n-1}(t)\right] \qquad (3)$$

In the next section, we use the Z-transform with given initial velocities to analyze eq. (3). In section 3, we use differential operators to obtain an expression for velocity of the $n^{th}$ car in the platoon. In section 4 the Laplace transform is used to derive the same results as in sections 2 and 3. In section 5, we discuss the linear car-following model with delay. Finally, we make concluding remarks in section 6. For a more detailed discussion of the car-following model see chapter 3 of Haberman[3].

## II. THE Z-TRANSFORM

The Z-transform for a sequence s(n) is given by:

$$X(s(\mathrm{n})) = \sum_{n=-\infty}^{\infty} s(\mathrm{n}) Z^{-n} \qquad (4)$$

A discussion of Z–transform and its properties can be found in anysignal processing textbook, such as Haddad and Parsons [4] orProakis and Manolakis[6]. We use the shifting property of Z-transform for delay signal $s$(n-k) given by:

$$X(s(\mathrm{n}-\mathrm{k})) = Z^{-k} X(s(\mathrm{n})) \quad (5)$$

The convolution of the two signals $s_1(n)$ and $s_2(n)$ is given by:

$$s_1(\mathrm{n}) * s_2(\mathrm{n}) = \sum_{k=-\infty}^{\infty} s_1(\mathrm{n}-\mathrm{k}) s_2(\mathrm{k}) \qquad (6)$$

An important property of the Z-transform is the convolution property i.e., the Z-Transform of the convolution is equal to the product of individual transforms, namely,

$$X(s_1(\mathrm{n}) * s_2(\mathrm{n})) = X(s_1(\mathrm{n})) \times X(s_2(\mathrm{n})) \quad (7)$$

The discrete unit step function is given by:

$$u(\mathrm{n}) = \begin{cases} 1 & \text{if n} \geq 0 \\ 0 & \text{if n} \leq 0 \end{cases}$$

If a signal $s$(n) is defined for n ≥ 0, we can extend the definition for all integers by multiplying $s$(n) by$u$(n). We use the shifting property and take the Z-transform of both sides of eq. (3).

Let X (z,t) denote the Z-transform of $v_n(t)$. We obtain:

$$\frac{dX(Z,t)}{dt} = -\lambda[X(Z,t) - Z^{-1}X(Z,t)] \qquad (8)$$

We obtain a separable differential equation:

$$\frac{dX(Z,t)}{X(Z,t)} = -\lambda(1 - Z^{-1})\, dt$$

By integrating both sides, we obtain:

$$\ln X(Z,t) = -\lambda(1 - Z^{-1})\, t + C$$

Exponentiating both sides and using the initial condition yields,

$$X(Z,t) = e^{-\lambda(1-Z^{-1})t} \bullet X(Z,0) \qquad (9)$$
$$= \mathrm{e}^{-\lambda t}(e^{\lambda Z^{-1} t}) \bullet X(Z,0)$$

The Maclaurian series for exponential yields,

$$X(Z,t) = e^{-\lambda t}\left[\sum_{n=0}^{\infty} \frac{(\lambda t)^n Z^{-n}}{n!}\right]\left[X(Z,0)\right] \qquad (10)$$

Note that $X(Z,0)$ is the Z-transform of initial velocities, namely,

$$X(Z,0) = \sum_{0}^{\infty} v_n(0) Z^{-\mathrm{n}}$$

Using convolution theorem to take inverse of eq. (10) we obtain:

$$v_n(t) = e^{-\lambda t}\, \frac{(\lambda t)^n}{n!} * v_n(0) \qquad (11)$$

Now use the definition of convolution to obtain $v_n(t)$,namely,

$$v_n(t) = e^{-\lambda t}\left[v_n(0) + \frac{(\lambda t)^1}{1!}.v_{n-1}(0) + \frac{(\lambda t)^2}{2!}.v_{n-2}(t) + ......\frac{(\lambda t)^n}{n!} v_0(0)\right]$$
$$= e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} v_{n-k}(0) \qquad (12)$$

Eq. (12) is consistent with the literature [2]. It is well known that the Fourier transform is obtained by evaluating the Z-transform evaluated on a unit circle. Thus, the above results can be analyzed using the Fourier transform.

## III. DIFFERENTIAL OPERATOR

Consider equation (3). Assume that the leading vehicle has constant velocity $v_0$. Let $V_n$ (t) denote relative velocity of the nth car with respect to the leading vehicle. That is, $V_n(t) = v_n(t) - v_0(t)$, Eq. (3) reduces to:

$$\frac{dV_n}{dt} = -\lambda(V_n(t) - V_{n-1}(t)) \qquad (13)$$

Now use differential operator D to obtain:

$$(\mathrm{D} + \lambda\mathrm{I})V_n = \lambda V_{n-1}(\mathrm{t}) \qquad (14)$$

Repeated application of eq. (14) implies that:

$$(\mathrm{D} + \lambda\mathrm{I})^n V_n = \lambda^n V_0 = 0 \qquad (15)$$

The solution to the differential eq. (15) is obtained by:

$$V_n(t) = e^{-\lambda t} \sum_{k=0}^{n-1} C_k t^k \qquad (16)$$

Use initial conditions for velocities to obtain:

$$v_n(t) = e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} v_{n-k}(0) \qquad (17)$$

Using discrete convolution, eq. (17) reduces to:

$$v_n(t) = e^{-\lambda t}\left[\frac{(\lambda t)^k}{k!} * v_{n-k}(0)\right] \qquad (18)$$

We can also use induction to prove eq. (17),as follows:

$$\text{For } \mathrm{n} = 1, \quad v_1(\mathrm{t}) = e^{-\lambda t} v_1(0) \qquad (19)$$

But eq. (19) is the solution to the differential eq. (20), namely,

$$\frac{dv_1}{dt} + \lambda v_1 = v_0 = 0 \qquad (20)$$

Now we assume eq. (17) for induction hypothesis and show the formula for
$v_{n+1}$. Note that:

$$\frac{dv_{n+1}}{dt} + \lambda v_{n+1} = \lambda v_n = \lambda\, e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} v_{n-k}(0) \quad (21)$$

Multiply both sides of eq. (21) by integrating factor $e^{\lambda t}$ to obtain:

$$e^{\lambda t} \frac{dv_{n+1}}{dt} + \lambda e^{kt} v_{n+1} = \lambda \sum_{k=0}^{n-1} \frac{(\lambda t)^{kt}}{k!} v_{n-k}(0) \qquad (22)$$

$$\frac{d\, e^{\lambda t} v_{n+1}}{dt} = \sum_{k=0}^{n-1} \frac{\lambda^{k+1}}{k!} t^k v_{n-1}(0) \qquad (23)$$

Integratebothsidesof eq. (23) and use initial condition $v_{n+1}(0)$ to obtain:

$$e^{\lambda t} v_{n+1}(t) = \sum_{k=0}^{n-1} \frac{(\lambda t)^{k+1}}{(k+1)!} v_{n-k}(0) + v_{n+1}(0) \qquad (24)$$

Let j=k+1

$$e^{\lambda t} v_{n+1}(t) = \sum_{j=1}^{n} \frac{(\lambda t)^j}{(j)!} v_{n-(j-1)}(0) + v_{n+1}(0) \qquad (25)$$

The right hand side of eq. (25) is equal to:

$$\sum_{j=0}^{n} \frac{(\lambda t)^j}{j!} v_{n+1-j}(0)$$

Change the index of summation by k and multiply both sides of eq. (25) by $e^{\lambda t}$to obtain:

$$v_{n+1}(t) = e^{-\lambda t} \sum_{k=0}^{n+1-1} \frac{(\lambda t)^k}{k!} v_{n+1-k}(0) \qquad (26)$$

This completes the proof of induction.

## IV. LAPLACE TRANSFORM

Recall that eq. (13) is a differential equation for velocity of the n[th] car in the platoon. A full discussion of Laplace transform is given in Ghandehari and Ardekani [2]. Take the Laplace transform of both sides of eq. (13) to obtain:

$$s\overline{v_n} - v_n(0) = -\lambda \overline{v_n} + \lambda \overline{v_{n-1}} \qquad (27)$$

$$\overline{v_n} = \frac{\lambda}{s+\lambda} \overline{v_{n-1}} + \frac{1}{s+\lambda} v_n(0) \qquad (28)$$

Apply eq. (28) recursively to obtain:

$$\overline{v_n} = \left(\frac{\lambda}{s+\lambda}\right)^n \overline{v_0} + \sum_{k=1}^{n-1} \left(\frac{\lambda}{s+\lambda}\right)^{n-k} v_k(0) \qquad (29)$$

The shifting property of Laplace transform is given by:

$$L(e^{-\lambda t} f(t)) = F(s+\lambda) \qquad (30)$$

In the above *F(s) = L(f(t))*. Also recall that:

$$L(t^k) = \frac{k!}{s^{k+1}} \qquad (31)$$

Use properties in eqs. (30) and (31) to take inverse of eq. (29). Also use summation notation to obtain:

$$v_n(t) = e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} v_{n-k}(0) \qquad (32)$$

This is the same as eq. (18). Use the assumption that the leading vehicle is moving with constant speed $v_o$to obtain:

$$v_n(t) = v_0 + e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} (v_{n-k}(0) - v_0) \qquad (33)$$

Finally, using the notation for convolution yields:

$$v_n(t) = v_0 + e^{-\lambda t} \left[ \frac{(\lambda t)^k}{k!} * (v_{n-k}(0) - v_0) \right] \qquad (34)$$

## V. INTRODUCING TIME DELAY

Assume *T* is the time delay representing the driver's reaction time to adjust speed. The mean value theorem implies that,

$$v_n(t+T) = v_n(t) + T\, v_n'(t) \qquad (35)$$

Use eqs. (1) and (35) and simplify to obtain the second order differential eq. (36), namely,

$$T\frac{d^2(v_n(t))}{dt^2} + \frac{dv_n}{dt} + \lambda v_n(t) = \lambda v_{n-1}(t) \qquad (36)$$

Consider the differential operator $T\ D^2 + D + \lambda I$. Apply the differential operator on eq. (36) recursively to obtain:

$$(TD^2 + D + \lambda I)^n\ v_n(t) = \lambda v_0(t) \qquad (37)$$

Let $r_1$ and $r_2$ denote roots of the characteristic eq. (38), namely,

$$(Tr^2 + r + \lambda) = 0 \qquad (38)$$

That is,

$$r = \frac{-1 \pm \sqrt{1 - 4\lambda T}}{2T} \qquad (39)$$

Assume the leading vehicle has constant speed $v_0$. The general solution of eq. (37) is then obtained by adding the homogenous and a particular solution $v_0$ to yield:

$$v_n(t) = v_0 + e^{r_1 t}\sum_{k=0}^{n-1} a_k t^k + e^{r_2 t}\sum_{k=0}^{n-1} b_k t^k \qquad (40)$$

In order to solve for constants in eq. (40), we need two sets of initial conditions (initial velocities as well as initial accelerations). If we only have initial velocities we have a perturbation problem with reaction time $T$ as a parameter. It will be interesting to see if the perturbation problem is regular or singular. In order to have real solution for eq. (40) we have the condition:

$$1 - 4\lambda T \geq 0.$$

That is

$$T \leq (1/4\lambda) \qquad (41)$$

If the direction of inequality is reserved then we have oscillatory solutions. With the assumption that $v_0$ is constant, the limit $v_n(t)$ as $t = \infty$ is equal to $v_0$ . This corresponds to practice: If the leading vehicle has constant speed, eventually all other vehicles reach the same constant speed.

We can also take the speed as $n = \infty$ .This will show that the perturbation in a leading vehicle will not have an effect on the vehicles too far in the back of the moving queue.

## VI. CONCLUSIONS

Methods from operational calculus have been used to analyze the linear car-following model. Transform methods as well as differential operators are used, to get the same results for the linear car-following model. Asymptotic analysis is performed to analyze the long-run behavior as well as the behavior for a very long string of vehicles.

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